The first nine problems are about the matrices,,,.

**1.5.1**. The 2 by 2 matrix in Example 1 has eigenvalues 1 and 3 in . Its unit eigenvectors and are the columns of . Multiply to recover .

**Sol**.

**1.5.2**. When you multiply the eigenvector by , the first row will produce a multiple of . Find that multiplier by a double-angle formula for : . Then \_\_\_ .

**Sol**. .

**1.5.3**. In MATLAB, construct and then its eigenvalues by . That column should be . Verify that agrees with 2 \* ones(5, 1) - 2 \* cos([1 : 5] \* pi/6)' .

**Sol**.

>> K=toeplitz([2,-1,0,0,0]); e=eig(K)

e =

0.27

1.00

2.00

3.00

3.73

>> 2 \* ones(5, 1) - 2 \* cos([1 : 5] \* pi/6)' -e

ans =

0.00

-0.00

0

-0.00

0

**1.5.4**. Continue the above question to find an eigenvector matrixby [Q,E]=eig(K).

The Discrete Sine Transform DST=Q\*diag([-1,-1,1,-1,1]) starts each column with a positive entry.

The matrix JK=[1:5 ]'\*[1:5] has entriestimes.

Verify that DST agrees with sin(JK\*pi/6)/sqrt(3) , and test.

**Sol**. >> K=toeplitz([2,-1,0,0,0]); [Q,E]=eig(K); DST=Q\*diag([-1,-1,1,-1,1]); JK=[1:5 ]'\*[1:5]; max(max(DST-sin(JK\*pi/6)/sqrt(3)))

ans =

3.3307e-16

**1.5.5**. Constructand [Q,E]=eig(B) with B(1,1)=1 and B(6,6)=1.

Verify that E=diag(e) with eigenvalues 2\*ones(1,6)-2\*cos([0:5]\* pi/6) in e.

How do you adjust Q to produce the (highly important) Discrete Cosine Transform with entries DCT=cos([.5:5.5]'\*[0:5]\*pi/6)/sqrt(3) ?

**Sol**. >> B=toeplitz([2,-1,0,0,0,0]); B(1,1)=1; B(6,6)=1; [Q,E]=eig(B); diag(E)-(2\*ones(6,1)-2\*cos([0:5]'\*pi/6))

ans =

1.0e-15 \*

0.2348

0.3886

0.2220

0.6661

-0.4441

0

Change the norm of the first column of Q toand make every first element in each column positive, and in this way Q becomes DCT.

**1.5.6**. The free-fixed matrixhas T(1,1)=1. Check that its eigenvalues are.

The matrix cos([.5:5.5 ]'\*[.5:5.5]\*pi/6.5)/sqrt(3.25) should contain its unit eigenvectors. Compute Q’\*Q and Q’\*T\*Q .

**Sol**. >> T=toeplitz([2,-1,zeros(1,4)]); T(1,1)=1; max(eig(T)-(2-2\*cos(([1:6]'-1/2)\*pi/6.5)))

ans =

0

>> Q=cos([.5:5.5 ]'\*[.5:5.5]\*pi/6.5)/sqrt(3.25); Q'\*Q

ans =

1.0000 0.0000 0.0000 -0.0000 0.0000 0.0000

0.0000 1.0000 0.0000 0.0000 0 0.0000

0.0000 0.0000 1.0000 0 0.0000 0.0000

-0.0000 0.0000 0 1.0000 -0.0000 0.0000

0.0000 0 0.0000 -0.0000 1.0000 -0.0000

0.0000 0.0000 0.0000 0.0000 -0.0000 1.0000

>> Q'\*T\*Q

ans =

0.0581 -0.0000 0.0000 -0.0000 0.0000 -0.0000

-0.0000 0.5030 0.0000 -0.0000 0.0000 0.0000

0.0000 0 1.2908 -0.0000 -0.0000 0.0000

-0.0000 -0.0000 -0.0000 2.2411 -0.0000 0.0000

0.0000 0 0 -0.0000 3.1361 -0.0000

-0.0000 0.0000 0.0000 0.0000 -0.0000 3.7709

**1.5.7**. The columns of the Fourier matrixare eigenvectors of the circulant matrix. But [Q,E]=eig(C) does not produce. What combinations of the columns ofgive the columns of? Notice the double eigenvalue in.

**Sol**. and enlarge each column to norm 2, and appears.

**1.5.8**. Show that the eigenvalues of add to the trace .

**Sol**.

**1.5.9**. andhave the same nonzero eigenvalues because they come from the samebackward difference. Show that and. The eigenvalues ofare the squared singular valuesofin 1.7.

**Sol**.

**1.5.10**. Factor these two matrices into . Check that and and

**Sol**.

**1.5.11**. If then . The eigenvectors of are (the same columns of )(different vectors) .

**Sol**. ,

**1.5.12**. Ifhaswith eigenvectorandwith, useto find. No other matrix has the same's and's.

**Sol**.

**1.5.13**. Suppose . What is the eigenvalue matrix for ? What is the eigenvector matrix? Check that .

**Sol**. , where eigenvectors keep the same, and each eigenvalues is added by 2.

**1.5.14**. If the columns of(eigenvectors of) are linearly independent, then

(a) is invertible (b) is diagonalizable (c) is invertible

**Sol**. All are true.

**1.5.15**. The matrixis not diagonalizable because the rank ofis \_\_\_. only has one line of eigenvector. Which entries could you change to makediagonalizable, with two eigenvectors?

**Sol**. is not diagonalizable because the rank ofis 1.

Any one of the entries could be changed to makediagonalizable with two eigenvectors.

**1.5.16**. approaches the zero matrix asif and only if everyhas absolute value less than \_\_\_. Which of these matrices has? and and .

**Sol**. approaches the zero matrix asif and only if everyhas absolute value less than 1.

has eigenvaluesand, so it has no

has eigenvaluesand, so it has

has eigenvalues,and, so it has no

**1.5.17**. Findandto diagonalizein Problem 16. What isfor these? and and

**Sol**.

**1.5.18**. Diagonalizeand computeto prove this formula for: has

**Sol**.

**1.5.19**. Diagonalizeand computeto show howinvolvesand: has

**Sol**.

**1.5.20**. Suppose. Take determinants to prove thatproduct of's. This quick proof only works whenis \_\_\_ .

**Sol**. . This quick proof only works whenis diagonal.

**1.5.21**. Show that, by adding the diagonal entries ofand: and.

Chooseand. Thenhas the same trace as, so the trace is the sum of the eigenvalues.

**Sol**.

**1.5.22**. Substituteinto the productand explain whyproduces the zero matrix. We are substitutingforin the polynomial. The Cayley-Hamilton Theorem says that(true even ifis not diagonalizable).

**Sol**.

**1.5.23**. Find's and's so thatsolves. What combinationstarts from?

**Sol**. .

**1.5.24**. Findto change the scalar equationinto a vector equation for. What are the eigenvalues of? Findandalso by substitutinginto.

**Sol**.

Also, plugintoto get

**1.5.25**. The rabbit and wolf populations show fast growth of rabbits (from) but loss to wolves (from). Findand its eigenvalues and eigenvectors:and. If, what are the populations at time? After a long time, is the ratio of rabbits to wolves 1 to 2 or is it 2 to 1 ?

**Sol**.

After a long time, , so the ratio of rabbits to wolves is 2 to 1

**1.5.26**. Substituteintoto show thatis a repeated root. This is trouble; we need a second solution after. The matrix equation is. Show that this matrix hasand only one line of eigenvectors. Trouble here too. Show that the second solution is.

**Sol**.

If,

And

**1.5.27**. Explain whyandhave the same eigenvalues. Show thatis always an eigenvalue whenis a Markov matrix, because each row ofadds to 1 and the vector \_\_\_ is an eigenvector of.

**Sol**. with the same eigenvalues.

**1.5.28**. Find the eigenvalues and unit eigenvectors ofand, and check the trace:

**Sol**.

**1.5.29**. Here is a quick "proof" that the eigenvalues of all real matrices are real: gives so is real. Find the flaw in this reasoning – a hidden assumption that is not justified.

**Sol**. might not be real, somight not be real.

**1.5.30**. Find all 2 by 2 matrices that are orthogonal and also symmetric. Which two numbers can be eigenvalues of these matrices?

**Sol**.

**1.5.31**. To find the eigenfunction, we could putin the differential equation. Then givesor. The complete solutionhasbecause. That simplifiesto a sine function: . yields. Thenmust be a multiple of, andas before. Repeat these steps forand also.

**Sol**.

**1.5.32**. Suppose eigshow follows and for these six matrices. How many real eigen-vectors? When does go around in the opposite direction from ?

**Sol**. has 2 real eigen-vectors. has 2 real eigen-vectors. go opposite at.

has 2 real eigen-vectors.go opposite at. has no real eigen-vector.

has 2 real eigen-vectors. has only 1 real eigen-vector.

**1.5.33**. Scary MATLAB shows what can happen when roundoff destroys symmetry: A=[1,1,1,1,1;1:5]'; B=A'\*A; P=A\*inv(B)\*A'; [Q,E]=eig(P); B is exactly symmetric. The projection P should be symmetric, but isn't. From Q'\*Q show that two eigenvectors of P fail badly to have inner product 0.

**Sol**. >> A=[1,1,1,1,1;1:5]'; B=A'\*A; P=A\*inv(B)\*A'; [Q,E]=eig(P);

>> Q'\*Q

ans =

1.0000 0.2944 0.0000 0.0000 0

0.2944 1.0000 0.0000 0.0000 0.0000

0.0000 0.0000 1.0000 -0.4016 0.6532

0.0000 0.0000 -0.4016 1.0000 -0.6143

0 0.0000 0.6532 -0.6143 1.0000